Chapter 14

Project Planning Using PERT/CPM

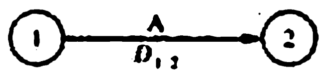
PERT/CPM

A project is a set of activities to be performed in a specific sequence to completion. An activity is a task to be executed using time and resources. The objective of project management is to minimize the total project time subject to the resource constraints. The two techniques widely used in project management are CPM (Critical Path Method) and PERT (Project/Program Evaluation and Review Technique). Although the two terms PERT and CPM are used interchangeably today, historically CPM was based on deterministic times while PERT was based on probabilistic times. In this chapter, project scheduling through PERT/CPM is discussed in the following order: Construction of the Network (Arrow) Diagram; Critical Path Computations for CPM; Critical Path Computations for PERT; and Project Time vs Project Cost.

CONSTRUCTION OF THE NETWORK (ARROW) DIAGRAM

The network diagram represents a project in that it shows the precedence relationships of the activities of the project along with activity times. The activities, which consume time and resources, are represented by "arrows." The precedence relationships of the activities are indicated through "events" (nodes). Events are just points in time, represented by circles. They do not consume any resources; they signify the beginning of some activities and the ending of some other activities.

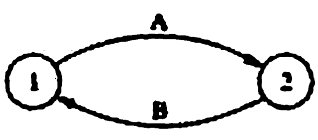
Consider the following diagram in which an activity (i, j) with duration $D_{i,j}$ is represented by an arrow between two events or nodes i (tail) and j (head). Usually the activities are named by letters (such as A, B, etc.), while the events are denoted by numbers (such as 1, 2, etc.).



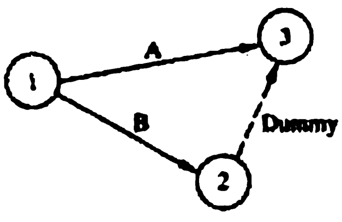
Sometimes in a network diagram it is necessary to use dummy activities, which consume no time or resources, represented by dotted arrows.

The purpose of the dummy activity can be one of the following:

(*a*) to represent an activity *uniquely* as below:

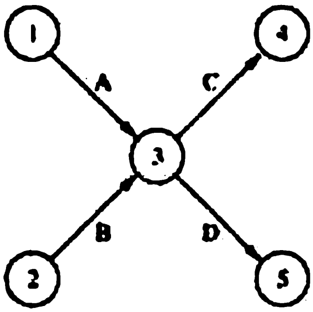


Both activities A and B are represented by events (1, 2).

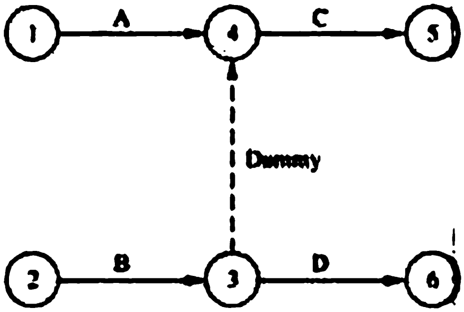


Activity A = events (1, 3)
Activity B = events (1, 2)

(*b*) to represent precedence relationships *exactly* as below:



Both activities C and D are individually preceded by both activities A and B.

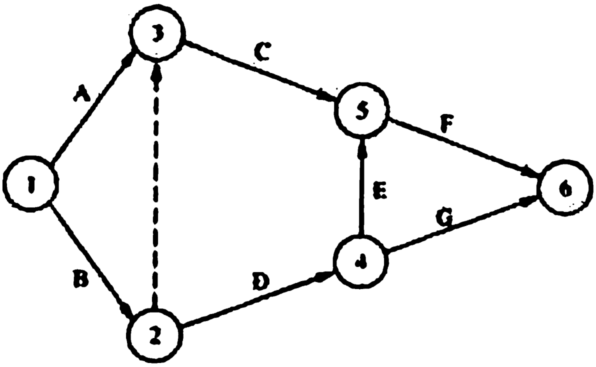


Activity C is preceded by both A and B. Activity D is preceded by B only.

Example 14.1 Construct a network diagram for a project consisting of the following activities:

Activity	Immediate Predecessor(s)
A	-
B	-
C	A, B
D	B
E	D
F	C, E
G	D

F and G are the terminal activities of the project.



CRITICAL PATH COMPUTATIONS FOR CPM

Through computations performed on the network (arrow) diagram (chart), CPM provides the following:

(*a*) start and completion times for each event

(*b*) critical and noncritical activities

(*c*) total float and free float times for activities.

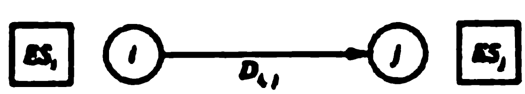
Example 14.2 Suppose the project in Example 14.1 has the following activity times:

Activity	Time (Days)
A	3
B	4
C	5
D	6
E	7
F	8
G	9

(*a*) Find the critical path.
(*b*) What is the project completion time?
(*c*) Compute the total floats (slacks) and free floats for the activities.

The critical path computations are performed in two passes—forward and backward. In the forward pass, starting with a time of 0 for the first event, the computations proceed from left to right up to the final event. The forward pass computations provide the earliest start (ES) times for the events. These times are entered in squares in the immediate vicinities of the corresponding events.

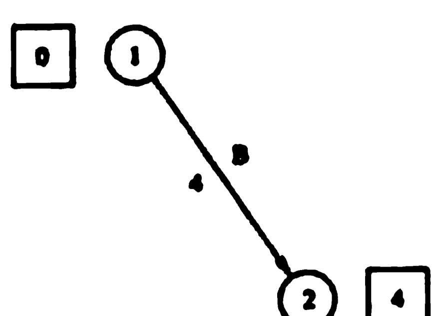
For any activity (i, j), let ES_i denote the earliest start time of event i.



Then
$$ES_j = ES_i + D_{i,j}$$

In Example 14.2, consider activity A (1, 2).

$$ES_2 = ES_1 + D_{1,2} = 0 + 4 = 4$$



If more than one activity enters an event, the earliest start time for that event is computed as follows:

$$ES_j = \max_i \{ES_i + D_{i,j}\}, \quad \text{for all } i \text{ entering into } j.$$

This is because the event cannot start until the entering activities are completed.

In Example 14.2, consider event 3. The two entering activities into event 3 are A (1, 3) and Dummy (2, 3).

$$ES_3 = \max_i \{ES_i + D_{i,3}\}, \quad i = 1, 2$$

$$= \max \{ES_1 + D_{1,3}, ES_2 + D_{2,3}\}$$

$$= \max \{0 + 3, 4 + 0\} = 4$$

Proceeding in a similar fashion, the earliest start times for all events are computed as shown below:

$$ES_4 = ES_2 + D_{2,4} = 4 + 6 = 10$$

$$ES_5 = \max_i \{ES_i + D_{i,5}\}, \quad i = 3, 4$$

$$= \max \{ES_3 + D_{3,5}, ES_4 + D_{4,5}\} = \max \{4 + 5, 10 + 7\} = 17$$

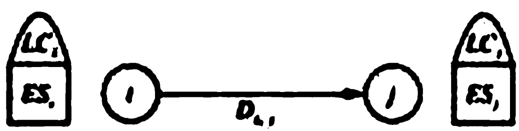
$$ES_6 = \max_i \{ES_i + D_{i,6}\}, i = 4, 5$$

$$= \max \{ES_4 + D_{4,6}, ES_5 + D_{5,6}\} = \max \{10 + 9, 17 + 8\} = 25$$

This ends the forward pass computations, giving a project completion time of 25 days.

In the backward pass, starting with the final node, the computations proceed from right to left up to the beginning event. The backward pass computations provide the latest completion (LC) times for the events. These times are entered in semicircles in the immediate vicinities of the corresponding events.

For any activity (i, j), let LC_i denote the latest completion time of event i.



Then

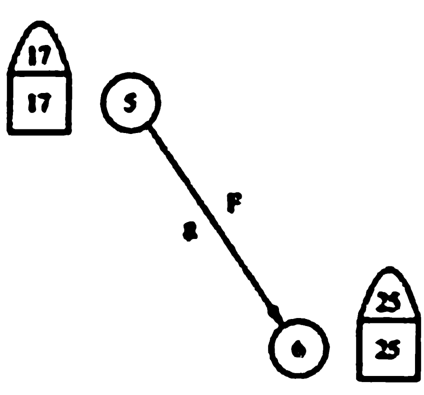
$$LC_i = LC_j - D_{i,j}$$

To get started, the latest completion time of the final event in the backward pass is assumed to be the earliest start time of that event in the forward pass.

In Example 14.2, consider activity F (5, 6).

$$LC_6 = ES_6 = 25$$

$$LC_5 = LC_6 - D_{5,6} = 25 - 8 = 17$$



If more than one activity leaves an event, the latest completion time for that event is computed as follows:

$$LC_i = \min_j \{LC_j - D_{i,j}\}, \quad \text{for all } j \text{ leaving from } i.$$

This will ensure progress in meeting the project completion time.

In Example 14.2, consider event 4. The two leaving activities from event 4 are E (4, 5) and G (4, 6).

$$LC_4 = \min_j \{LC_j - D_{4,j}\}, \quad j = 5, 6$$
$$= \min \{LC_5 - D_{4,5}, LC_6 - D_{4,6}\}$$
$$= \min \{17 - 7, 25 - 9\} = 10$$

Proceeding in a similar fashion, the latest completion times for all events are completed as shown below:

$$LC_3 = LC_5 - D_{3,5} = 17 - 5 = 12$$

$$LC_2 = \min_j \{LC_j - D_{2,j}\}, \quad j = 3, 4$$
$$= \min \{LC_3 - D_{2,3}, LC_4 - D_{2,4}\}$$
$$= \min \{12 - 0, 10 - 6\} = 4$$

$$LC_1 = \min_j \{LC_j - D_{1,j}\}, \quad j = 2, 3$$
$$= \min \{LC_2 - D_{1,2}, LC_3 - D_{1,3}\}$$
$$= \min \{4 - 4, 12 - 3\} = 0$$

This ends the backward pass computations, confirming the project start time of 0.

After completing the critical path (forward pass and backward pass) computations, the complete network diagram appears as in Fig. 14-1. This enables us to determine the critical activities of the project. An activity (i, j) is said to be critical, if and only if it satisfies all the conditions given below:

1. $ES_i = LC_i$
2. $ES_j = LC_j$
3. $ES_j - ES_i = LC_j - LC_i = D_{i,j}$

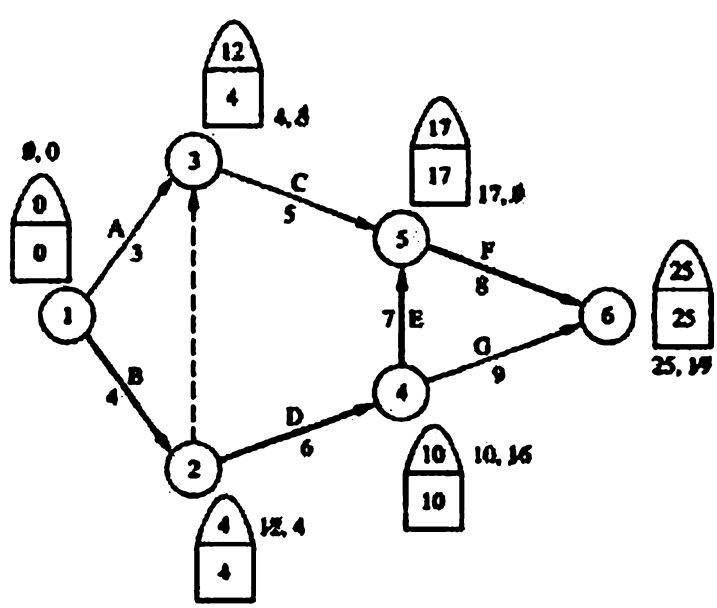


Fig. 14-1. Critical path by CPM

Application of the above conditions to Example 14.2 indicates the following activities to be critical: B, D, E, and F. In other words, activities B, D, E, and F form the critical path. In terms of events, 1-2-4-5-6 is the critical path. The project completion time is 25 days. It must be noted that a critical path is a continuous path starting with the first event and ending with the last event. It is shown in bold line. Also, in some cases, it is possible to have multiple critical paths. Now, we can define the slack or total float for an activity (i, j) as follows:

$$\text{Slack} = \text{Total Float} = TF_{i,j} = LC_j - ES_i - D_{i,j}$$

The total float for an activity is the difference between its maximum available time $(LC_j - ES_i)$ and its duration $(D_{i,j})$. It signifies the time by which an activity can be delayed without delaying the project. A zero slack for an activity indicates that it cannot be delayed without delaying the project and hence it is called a critical activity. On the other hand, a positive slack for an activity means that it can be delayed by the length of the slack without delaying the project and hence it is called a noncritical activity.

In Example 14.2, consider the total floats for activities C (3, 5) and D (2, 4):

$$\text{Activity C:} \quad TF_{3,5} = LC_5 - ES_3 - D_{3,5} = 17 - 4 - 5 = 8$$

Activity C has a positive slack indicating it is noncritical.

$$\text{Activity D:} \quad TF_{2,4} = LC_4 - ES_2 - D_{2,4} = 10 - 4 - 6 = 0$$

Activity D has a zero slack indicating it is critical.

Similarly the total floats for all the other activities in Example 14.2 can be computed.

There is yet another float called free float, which is useful in considering project time vs project cost. Free float for an activity (i, j) is the difference between its available time (based on earliest start times) and its duration. It is given as follows:

$$FF_{i,j} = ES_j - ES_i - D_{i,j}$$

In Example 14.2, consider the free floats for activities A (1, 3) and E (4, 5):

$$\text{Activity A:} \quad FF_{1,3} = ES_3 - ES_1 - D_{1,3} = 4 - 0 - 3 = 1$$
$$\text{Activity E:} \quad FF_{4,5} = ES_5 - ES_4 - D_{4,5} = 17 - 10 - 7 = 0$$

Similarly the free floats for all the other activities in Example 14.2 can be calculated.

The above calculations are summarized in Table 14-1.

Table 14-1 Critical path calculations including floats

Activity (i, j)	Duration $D_{i,j}$	ES_i	LC_j	ES_j	Total Float (Slack) $TF_{i,j} = LC_j - ES_i - D_{i,j}$	Critical	Free Float $FF_{i,j} = ES_j - ES_i - D_{i,j}$
A (1, 3)	3	0	12	4	12 − 0 − 3 = 9	-	4 − 0 − 3 = 1
B (1, 2)	4	0	4	4	4 − 0 − 4 = 0	Yes	4 − 0 − 4 = 0
C (3, 5)	5	4	17	17	17 − 4 − 5 = 8	-	17 − 4 − 5 = 8
D (2, 4)	6	4	10	10	10 − 4 − 6 = 0	Yes	10 − 4 − 6 = 0
E (4, 5)	7	10	17	17	17 − 10 − 7 = 0	Yes	17 − 10 − 7 = 0
F (5, 6)	8	17	25	25	25 − 17 − 8 = 0	Yes	25 − 17 − 8 = 0
G (4, 6)	9	10	25	25	25 − 10 − 9 = 6	-	25 − 10 − 9 = 6

CRITICAL PATH COMPUTATIONS FOR PERT

The network diagrams for PERT and CPM are identical except for the activity times. The time estimates for CPM are deterministic, while those for PERT are probabilistic. In PERT, each activity has the following three time estimates:

a = optimistic time estimate under the best of conditions
b = pessimistic time estimate under the worst of conditions
m = most likely (probable) time estimate under normal conditions

The probabilistic nature of the activity times is described by the beta distribution whose mean and variance are given below:

Mean: $E(D_{i,j}) = (a + b + 4m)/6$

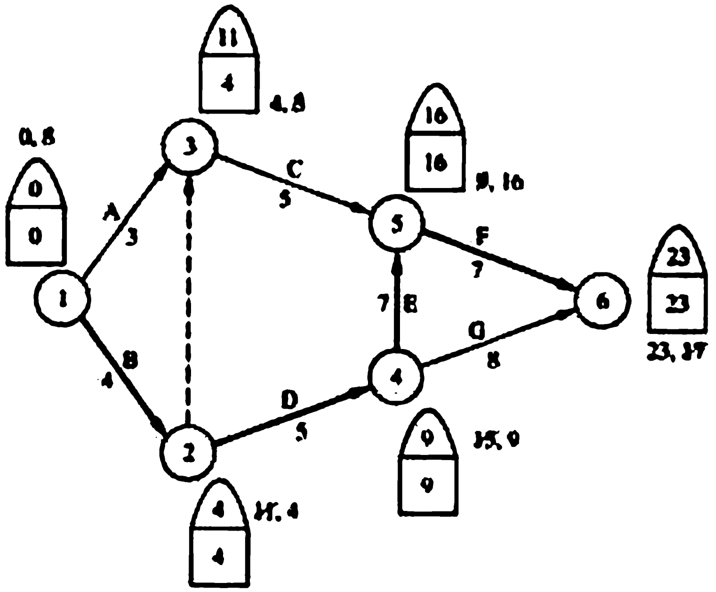
Variance: $\sigma^2_{i,j} = [(b - a)/6]^2$

Example 14.3 Suppose the following estimates of activity times (days) are provided for Example 14.1.

Activity	Optimistic (a)	Most Likely (m)	Pessimistic (b)
A	1	3	5
B	3	4	5
C	4	5	6
D	3	5	7
E	5	6	13
F	4	7	10
G	6	8	10

(*a*) Determine the expected completion time and variance for the project.
(*b*) What is the probability that the project will be completed within 20 days? 25 days?

Activity (i, j)	Expected Time $E(D_{i,j})$	Variance $\sigma^2_{i,j}$
A (1, 3)	3	0.4444
B (1, 2)	4	0.1111
C (3, 5)	5	0.1111
D (2, 4)	5	0.4444
E (4, 5)	7	1.7777
F (5, 6)	7	1.0000
G (4, 6)	8	0.4444



Critical path is B, D, E, F.

(*a*) Expected project completion time $= E(T) = E(T_B) + E(T_D) + E(T_E) + E(T_F) = 4 + 5 + 7 + 7 = 23$

$$\text{Project variance} = \sigma^2 = \sigma_B^2 + \sigma_D^2 + \sigma_E^2 + \sigma_F^2 = 0.1111 + 0.4444 + 1.7777 + 1 = 3.3332$$

(*b*) Probability that the project completion time $T \leq 20$ days:

$$K = 20$$

$$E(T) = 23$$

$$\sigma = \sqrt{3.3332} = 1.83$$

$$c = \frac{K - E(T)}{\sigma} = \frac{20 - 23}{1.83} = -1.64$$

$P(T \leq 20) = P(Z \leq C) = P(Z \leq -1.64) = 0.0505$ (from normal distribution tables)

Probability that the project completion time $T \leq 25$ days:

$$K = 25$$

$$E(T) = 23$$

$$\sigma = 1.83$$

$$C = \frac{K - E(T)}{\sigma} = \frac{25 - 23}{1.83} = 1.09$$

$P(T \leq 25) = P(Z \leq C) = P(Z \leq -1.09) = 0.8621$ (from normal distribution tables).

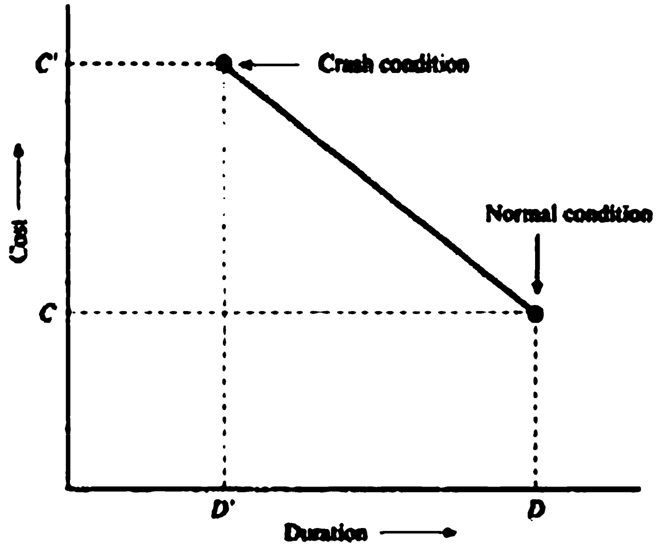
PROJECT TIME VS PROJECT COST

The "normal" time for an activity can be reduced by using increased resources. The limit beyond which an activity time cannot be shortened is known as the "crash limit." Let D and C represent the normal time (duration) and normal cost for an activity, while D' and C' denote the crash time (duration) and crash cost for the same activity. Then the "crash limit" for an activity is the difference between its normal time and its crash time.

$$\text{Crash Limit} = D - D'$$

Assuming a straight line cost–duration relationship,

$$\text{Slope} = (C' - C)/(D - D') = \text{Crash cost per unit time.}$$



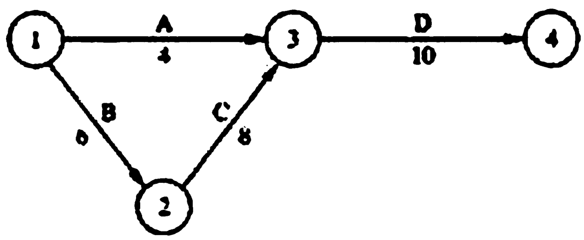
The project completion time can be reduced by reducing the normal times of critical activities. Reducing the critical activity with the minimum cost–duration slope will yield the minimum cost. This critical activity can be reduced up to the "crash limit."

This does not guarantee that the project time will also be reduced by the same length, since the above reduction may have led to a new critical path. To find whether a new critical path may occur, check whether a positive free float of any non-critical activity becomes zero. By reducing the duration of the critical activity by one time unit, compute the new free floats of the non-critical activities; check which ones have reduced their old positive free floats by one unit; of these, the one with the smallest old positive free float gives the positive free float limit. Thus for a critical activity,

$$\text{reduction limit} = \min\{\text{crash limit, positive free float limit}\}$$

Continue to proceed in the above fashion until all critical activities in the latest critical path are at their crash limits.

Example 14.4 Consider the following arrow diagram with activity times given in days:



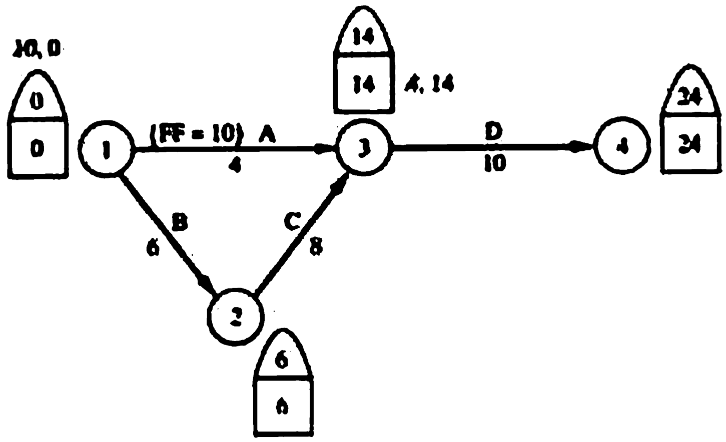
The normal and crash data for this project are as follows:

Activity	Normal Time (Days)	Crash Time (Days)	Normal Cost ($)	Crash Cost ($)
A	4	3	80	105
B	6	4	180	250
C	8	5	200	320
D	10	6	350	530

(*a*) Find the critical path.

(*b*) Find the project completion time and the corresponding cost.

(*c*) If we want to complete the project in 18 days, find the best crash time and cost.



(*a*) Critical path is B, C, D.

(*b*) Project completion time = 24 days
Project cost = 80 + 180 + 200 + 350 = $810

(*c*) From the given data, construct the following crash time–cost table.

Activity (i, j)	Crash Limit $(D - D')$	Crash Cost/Day $(C' - C)/(D - D')$
A (1, 3)	4 − 3 = 1	(105 − 80)/(4 − 3) = 25
B (1, 2)	6 − 4 = 2	(250 − 180)/(6 − 4) = 35
C (2, 3)	8 − 5 = 3	(320 − 200)/(8 − 5) = 40
D (3, 4)	10 − 6 = 4	(530 − 350)/(10 − 6) = 45

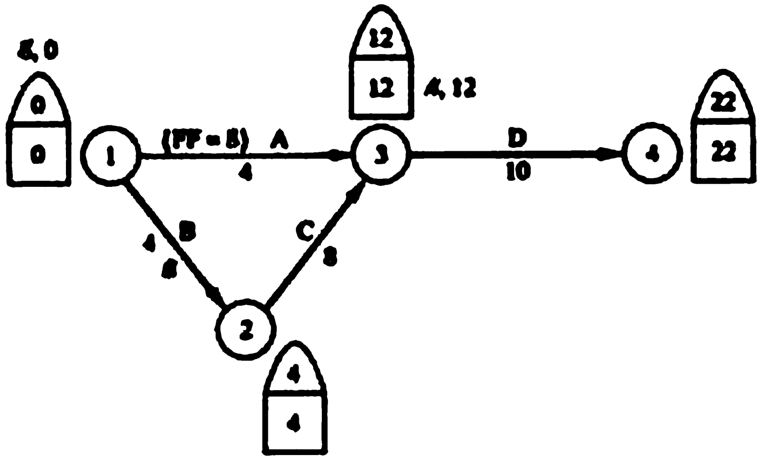
From the critical path calculations, we have the following information:

Activity (i, j)	A (1, 3)	B (1, 2)	C (2, 3)	D(3, 4)
Critical	-	yes	yes	yes
Free Float (FF)	10	-	-	-

Since the critical activity B has the lowest "crash cost per day," it becomes the first candidate for crash. The length by which B can be reduced is found as follows:

reduction limit = min {crash limit, positive FF limit} = min {2, 10} = 2

Hence, crash activity B by 2 days.



From the critical path calculations, we have the following information:

Critical path is still B, C, D

Project completion time = 22 days

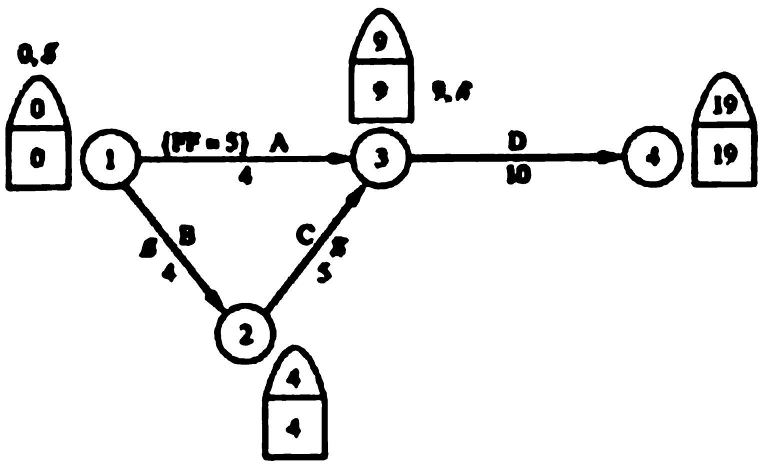
Project cost = 810 + (2)(35) = $880

Activity (i, j)	A (1, 3)	B (1, 2)	C (2, 3)	D (3, 4)
Critical	-	yes	yes	yes
Free Float (FF)	8	-	-	-

Since the crash limit for critical activity B is reached, consider critical activity C with the next lowest "crash cost per day" for crash. The length by which C can be reduced is found as follows:

$$\text{reduction limit} = \min\{\text{crash limit, positive FF limit}\} = \min\{3, 8\} = 3$$

Hence, crash activity C by 3 days.



From the critical path calculations, we have the following information:

Critical path is still B, C, D

Project completion time = 19 days

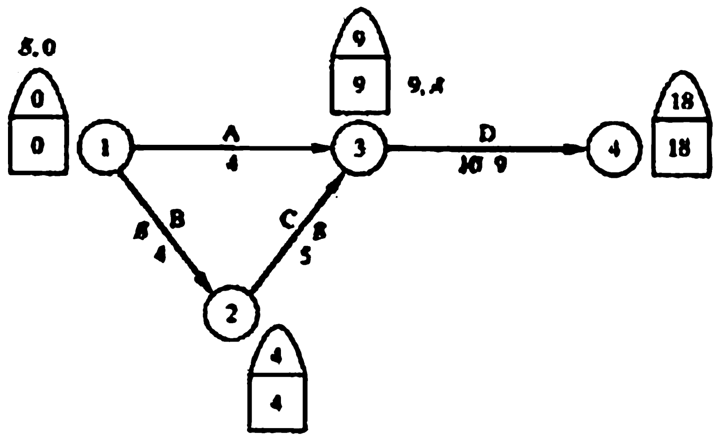
Project cost = 880 + (3)(40) = $1000

Activity (i, j)	A (1, 3)	B (1, 2)	C (2, 3)	D (3, 4)
Critical	-	yes	yes	yes
Free Float (FF)	5	-	-	-

Since the crash limit for critical activity C is reached, consider critical activity D with the next lowest "crash cost per day" for crash. The length by which D can be reduced is found as follows:

$$\text{reduction limit} = \min\{\text{crash limit, positive FF limit}\} = \min\{4, 5\} = 4$$

Although we can reduce D by 4 days, it is only necessary to reduce it by 1 day to reach our project completion goal of 18 days. (Note: the project completion time from the previous critical path calculations is 19 days.)



From the critical path calculations, we have the following information:

Critical path is still B, C, D
Projection completion time = 18 days
Project cost = 1000 + (1)(45) = $1045

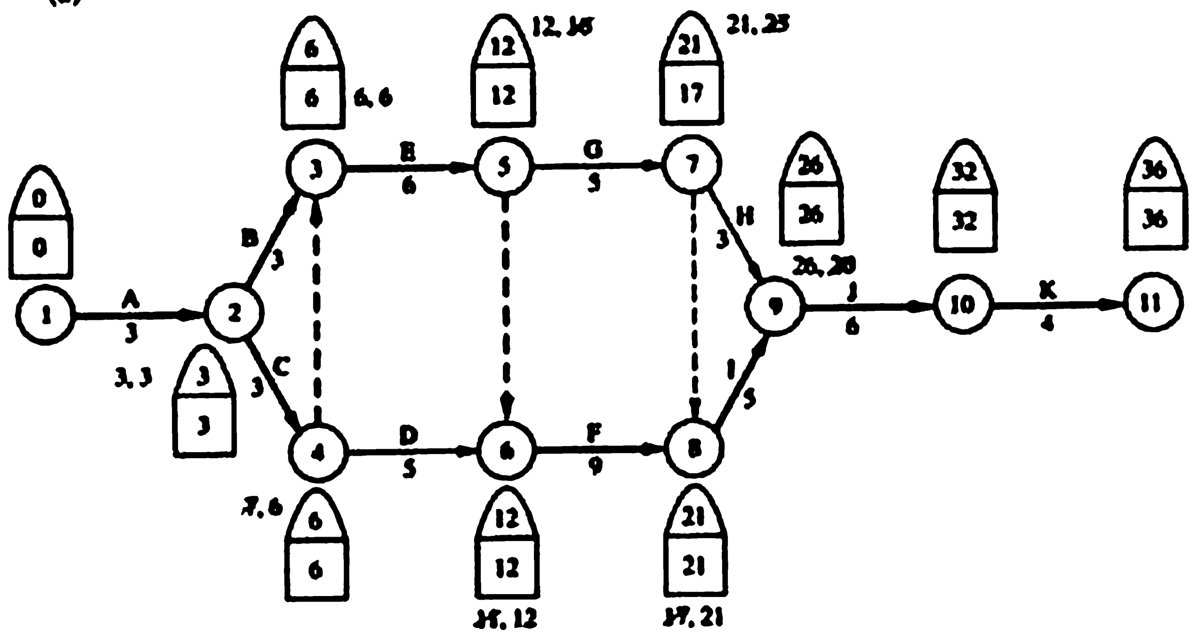
Solved Problems

14.1 The ABC Manufacturing Company is considering the construction of a new factory building. The following list shows the project activities, precedence relationships, and time estimates:

Activity	Description	Immediate Predecessor(s)	Time
A	Problem definition	–	3
B	Preliminary study of costs and constraints	A	3
C	Analysis of problems in existing building	A	3
D	Incorporation of requirements in new building	C	5
E	Detailed drawings of new building	B, C	6
F	Contractor building a prototype	D, E	9
G	Cost analysis	E	5
H	Engineers reviewing feasibility	G	3
I	Contractor building the factory	G, F	5
J	Building inspection	I, H	6
K	Final plant layout	J	4

(*a*) Develop a CPM network for this project.
(*b*) Identify the critical path.
(*c*) Compute the total and free floats for the activities.

(*a*)



(*b*) Critical paths are A, B, E, F, I, J, K and A, C, E, F, I, J, K.

(*c*)

Activity (i, j)	Duration $D_{i,j}$	ES_i	LC_j	ES_j	Total Float (Slack) $TF_{i,j} = LC_j - ES_i - D_{i,j}$	Critical	Free Float $FF_{i,j} = ES_j - ES_i - D_{i,j}$
A (1, 2)	3	0	3	3	3 − 0 − 3 = 0	Yes	3 − 0 − 3 = 0
B (2, 3)	3	3	6	6	6 − 3 − 3 = 0	Yes	6 − 3 − 3 = 0
C (2, 4)	3	3	6	6	6 − 3 − 3 = 0	Yes	6 − 3 − 3 = 0
D (4, 6)	5	6	12	12	12 − 6 − 5 = 1	-	12 − 6 − 5 = 1
E (3, 5)	6	6	12	12	12 − 6 − 6 = 0	Yes	12 − 6 − 6 = 0
F (6, 8)	9	12	21	21	21 − 12 − 9 = 0	Yes	21 − 12 − 9 = 0
G (5, 7)	5	12	21	17	21 − 12 − 5 = 4	-	17 − 12 − 5 = 0
H (7, 9)	3	17	26	26	26 − 17 − 3 = 6	-	26 − 17 − 3 = 6
I (8, 9)	5	21	26	26	26 − 21 − 5 = 0	Yes	26 − 21 − 5 = 0
J (9, 10)	6	26	32	32	32 − 26 − 6 = 0	Yes	32 − 26 − 6 = 0
K (10, 11)	4	32	36	36	36 − 32 − 4 = 0	Yes	36 − 32 − 4 = 0

14.2 An industrial project has the following data:

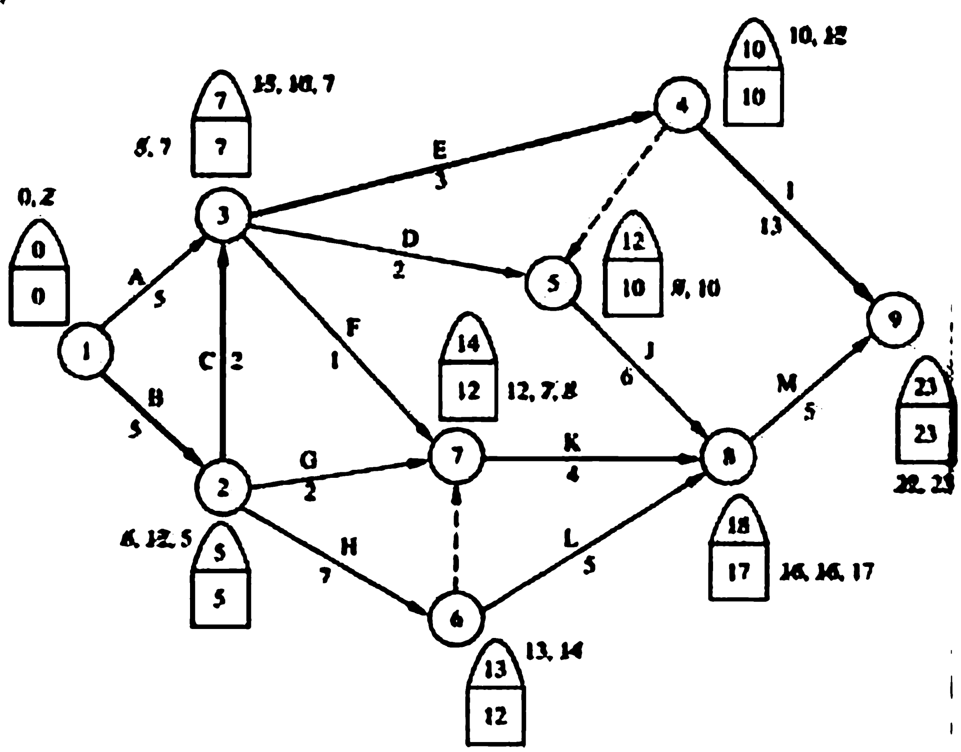
Activity	Immediate Predecessor(s)	Duration (Weeks)
A	–	5
B	-	5
C	B	2
D	A, C	2
E	A, C	3
F	A, C	1
G	B	2
H	B	7
I	E	13
J	E, D	6
K	F, G, H	4
L	H	5
M	J, K, L	5

I and M are the terminal activities of the project.

(*a*) Develop a network diagram and find the critical path.

(*b*) Compute the total and free floats for the activities.

(*a*)



Critical path is B, C, E, I.

(b)

Activity (i, j)	Duration $D_{i,j}$	ES_i	LC_j	ES_j	Total Float (Slack) $TF_{i,j} = LC_j - ES_i - D_{i,j}$	Critical	Free Float $FF_{i,j} = ES_j - ES_i - D_{i,j}$
A (1, 3)	5	0	7	7	7 − 0 − 5 = 2	-	7 − 0 − 5 = 2
B (1, 2)	5	0	5	5	5 − 0 − 5 = 0	Yes	5 − 0 − 5 = 0
C (2, 3)	2	5	7	7	7 − 5 − 2 = 0	Yes	7 − 5 − 2 = 0
D (3, 5)	2	7	12	10	12 − 7 − 2 = 3	-	10 − 7 − 2 = 1
E (3, 4)	3	7	10	10	10 − 7 − 3 = 0	Yes	10 − 7 − 3 = 0
F (3, 7)	1	7	14	12	14 − 7 − 1 = 6	-	12 − 7 − 1 = 4
G (2, 7)	2	5	14	12	14 − 5 − 2 = 7	-	12 − 5 − 2 = 5
H (2, 6)	7	5	13	12	13 − 5 − 7 = 1	-	12 − 5 − 7 = 0
I (4, 9)	13	10	23	23	23 − 10 − 13 = 0	Yes	23 − 10 − 13 = 0
J (5, 8)	6	10	18	17	18 − 10 − 6 = 2	-	17 − 10 − 6 = 1
K (7, 8)	4	12	18	17	18 − 12 − 4 = 2	-	17 − 12 − 4 = 1
L (6, 8)	5	12	18	17	18 − 12 − 5 = 1	-	17 − 12 − 5 = 0
M (8, 9)	5	17	23	23	23 − 17 − 5 = 1	-	23 − 17 − 5 = 1

14.3 Draw a PERT network diagram for a construction project with the activity information given below:

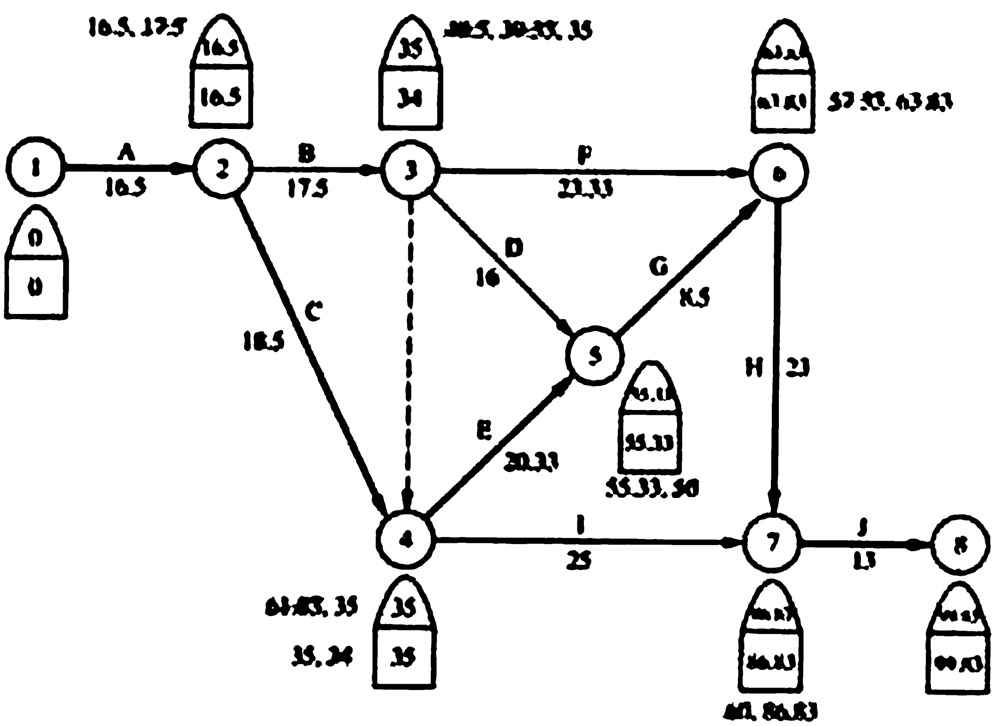
		Duration (Weeks)		
Activity	Immediate Predecessor(s)	Optimistic (a)	Most Likely (m)	Pessimistic (b)
A	-	7	16	28
B	A	4	19	25
C	A	10	16	37
D	B	7	13	37
E	B, C	13	19	33
F	B	19	22	33
G	D, E	4	7	19
H	F, G	13	19	49
I	B, C	13	25	37
J	I, H	7	13	19

(a) Identify the critical path.

(b) Determine the probability of completing the project in two years (104 weeks).

(a)

Activity (i, j)	Expected Time $E(D_{i,j})$	Variance $\sigma^2_{i,j}$
A (1, 2)	16.5	12.25
B (2, 3)	17.5	12.25
C (2, 4)	18.5	20.25
D (3, 5)	16	25
E (4, 5)	20.33	11.11
F (3, 6)	23.33	5.44
G (5, 6)	8.5	6.25
H (6, 7)	23	36
I (4, 7)	25	16
J (7, 8)	13	4



Critical path is A, C, E, G, H, J.

(b) Probability that the product completion time $T \leq 104$ weeks:

$$K = 104$$

$$E(T) = 99.83$$

$$\sigma^2 = \sigma_A^2 + \sigma_C^2 + \sigma_E^2 + \sigma_G^2 + \sigma_H^2 + \sigma_I^2$$

$$= 12.25 + 20.25 + 11.11 + 6.25 + 36 + 4 = 89.96$$

$$\sigma = \sqrt{89.96} = 9.48$$

$$C = \frac{K - E(T)}{\sigma} = \frac{104 - 99.83}{9.48} = 0.44$$

$$P(T \le 33) = P(z \le C) = P(z \le 0.44) = 0.17 \quad \text{(from normal distribution tables)}$$

14.4 The project of constructing a small bridge in Wilmington, Pennsylvania consists of 10 major activities. Information pertaining to the project is given below:

Activity	Optimistic (*a*)	Most Likely (*m*)	Pessimistic (*b*)
A	2	5	8
B	4	7	10
C	4	9	14
D	6	10	20
E	1	3	5
F	3	6	9
G	4	5	12
H	6	8	10

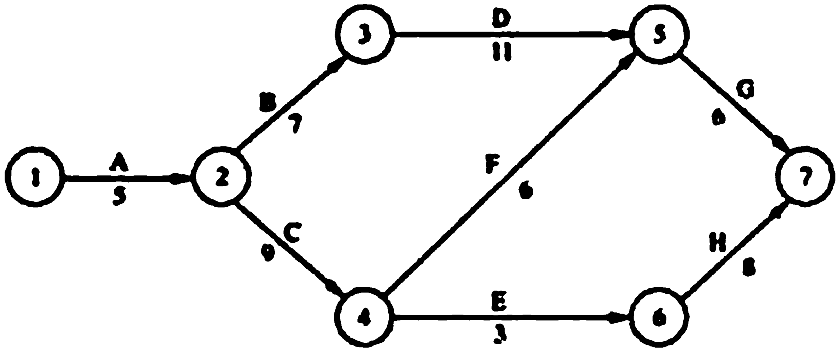
(*a*) Develop a PERT network for this project.

(*b*) Find the critical path.

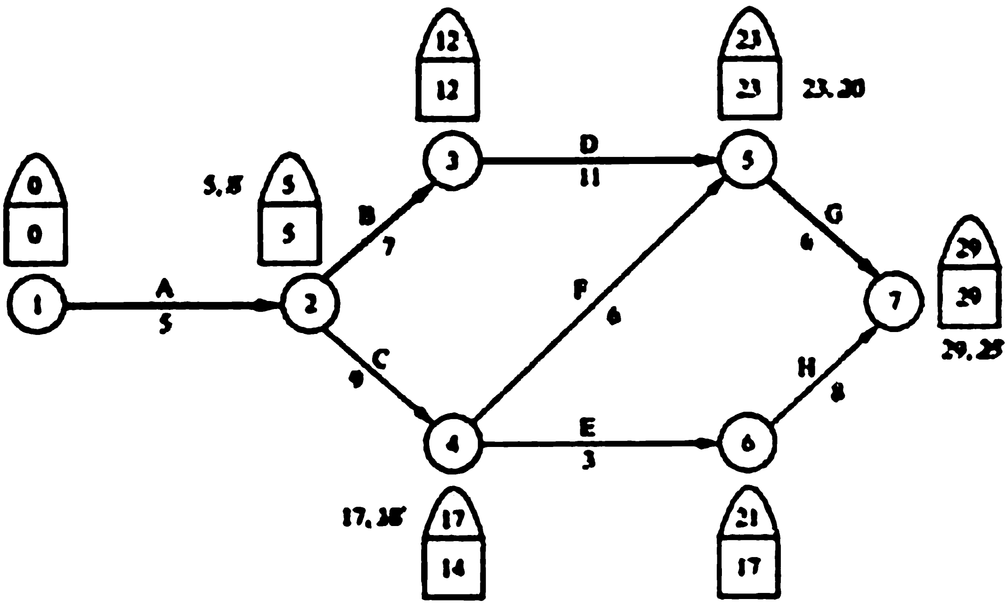
(*c*) Compute the probability of completing the project in 29 weeks.

(*a*)

Activity (i, j)	Expected Time $E(D_{i,j})$	Variance $\sigma_{i,j}^2$
A (1, 2)	5	1
B (2, 3)	7	1
C (2, 4)	9	2.78
D (3, 5)	11	5.44
E (4, 6)	3	0.44
F (4, 5)	6	1
G (5, 7)	6	1.78
H (6, 7)	8	0.44



(b)



Critical path is A, B, D, G.

(c) Probability that the project completion time $T \leq 36$ weeks:

$$K = 36$$

$$E(T) = 29$$

$$\sigma^2 = \sigma_A^2 + \sigma_B^2 + \sigma_D^2 + \sigma_G^2 = 1 + 1 + 5.44 + 1.78 = 9.22$$

$$\sigma = \sqrt{9.22} = 3.04$$

$$C = \frac{K - E(T)}{\sigma} = \frac{36 - 29}{3.04} = 2.30$$

$P(T \leq 36) = P(z \leq C) = P(z \leq 2.30) = 0.9893$ (from normal distribution tables)

14.5 Fusion Engineering Inc. is designing a new product for welding two different alloys. The company has limited time and resources to complete the project. The following activity information is available.

Activity	Immediate Predecessor(s)	Normal Time (Days)	Normal Cost ($)	Crash Cost/Day ($)	Crash Time (Days)
A	–	4	400	125	3
B	A	5	800	200	4
C	A	4	520	150	2
D	B	3	600	225	2
E	C	3	255	100	2
F	B, E	4	600	175	2

(*a*) Draw the project network.

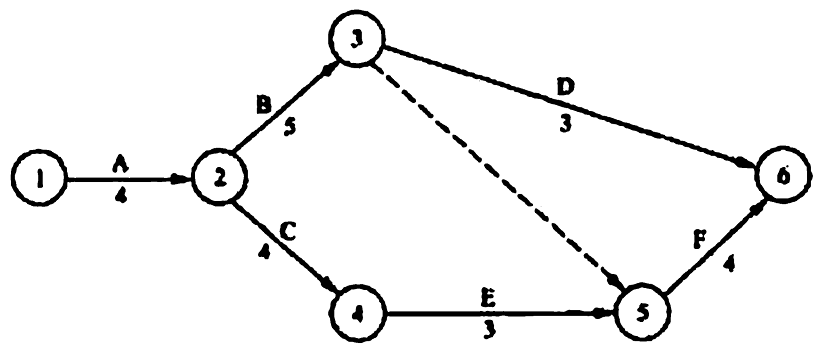
(*b*) Find the critical path.

(*c*) Find the project completion time and the corresponding cost.

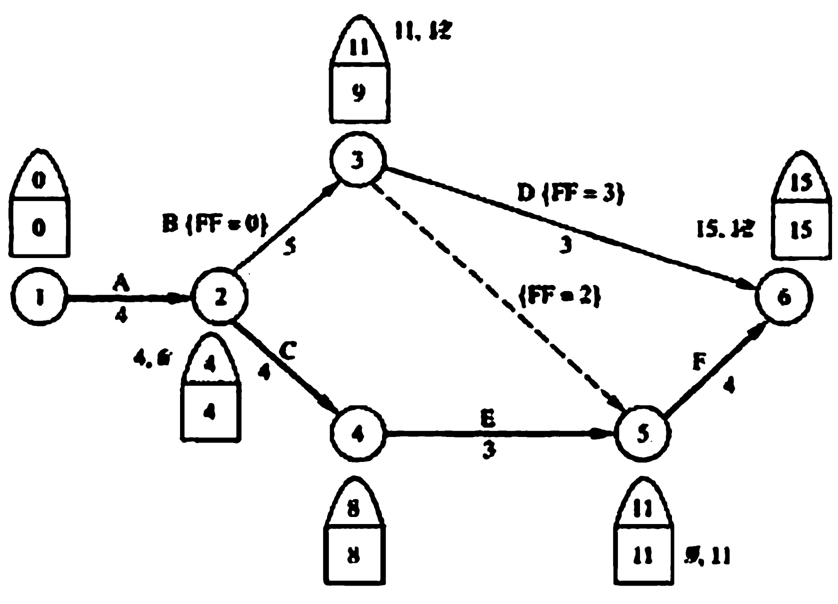
(*d*) What is the total cost, if the project deadline is 13 days?

(*e*) Assume the project deadline to be 10 days. The company has to bear $170 for each day of delay. Find the optimal number of days to crash the project.

(*a*)



(*b*)



Critical path is A, C, E, F.

(c) Project completion time = 15

$$\text{Project cost} = 400 + 800 + 520 + 600 + 255 + 600 = 3175$$

(d) From the given data, construct the following crash time–cost table.

Activity (*i*, *j*)	Crash Limit (*D* – *D*′) Days	Crash Cost/Day (Given) $
A (1, 2)	4 – 3 = 1	125
B (2, 3)	5 – 4 = 1	200
C (2, 4)	4 – 2 = 2	150
D (3, 6)	3 – 2 = 1	225
E (4, 5)	3 – 2 = 1	100
F (5, 6)	4 – 2 = 2	175

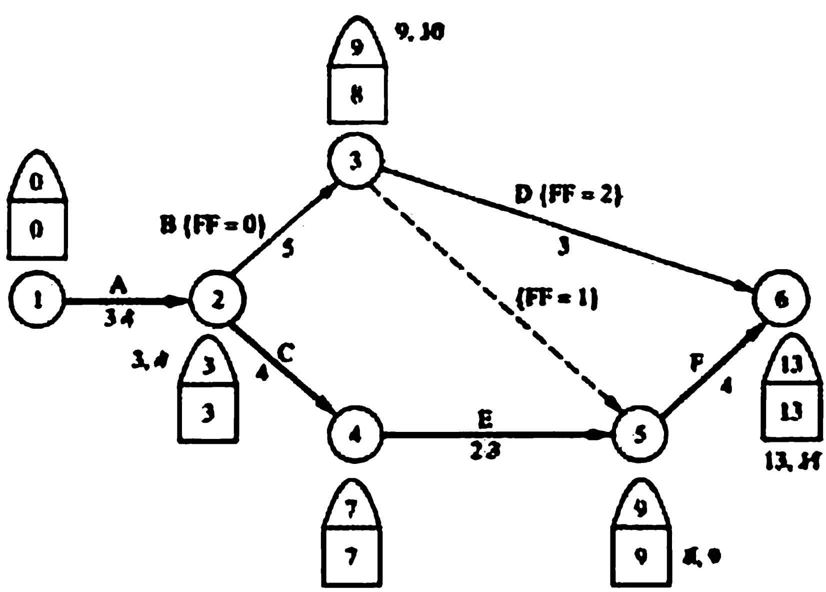
From the critical path calculations, we have the following information.

Activity (*i*, *j*)	A (1, 2)	B (2, 3)	C (2, 4)	D (3, 6)	E (4, 5)	F (5, 6)	Dummy (3, 5)
Critical	yes	-	yes	-	yes	yes	-
Free Float (FF)	-	0	-	3	-	-	2

Since the normal project completion time is 15 days and the required project completion time is 13 days, we have to crash one or more critical activities for a total of 2 days. The two lowest "crash cost per day" critical activities E and A have crash limits of 1 day each for a total of 2 days.

$$\text{reduction limit} = \min\ (\text{crash limit, positive FF limit}) = \min\ \{2, 2\} = 2$$

Hence, crash activities E and A by one day each.



From the critical path calculations, we have the following information.

Critical path is still A, C, E, F.

Project completion time = 13 days

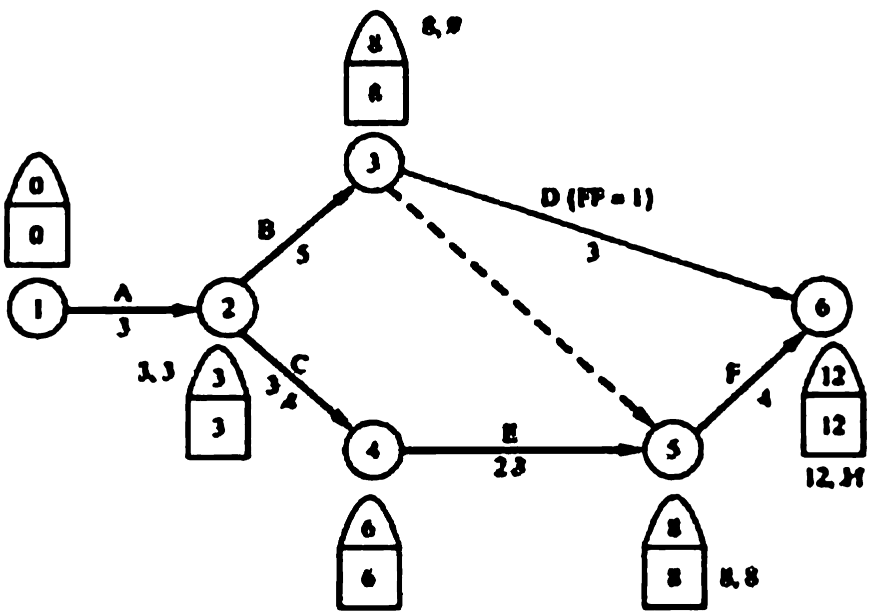
Project cost = 3175 + (1)(100) + (1)(125) = $3400

Activity (i, j)	A (1, 2)	B (2, 3)	C (2, 4)	D (3, 6)	E (4, 5)	F (5, 6)	Dummy (3, 5)
Critical	yes	-	yes	-	yes	yes	-
Free Float (FF)	-	0	-	2	-	-	1

(e) Since the above project completion time is 13 days and the new project deadline is 10 days, we will try to crash the project for a total of 3 days. Since the crash limits for critical activities E and A are reached, consider critical activity C with the next lowest "crash cost per day" for crash. The length by which C can be reached is found as follows:

reduction limit = min {crash limit, positive FF limit} = min {2, 1} = 1

Hence, crash activity C by one day.



From the critical path calculations, we have the following information.

There are two critical paths: A, C, E, F (old) and A, B, F (new)

Project completion time = 12 days

Project cost = 3400 + (1)(150) = $3550

Activity (i, j)	A (1, 2)	B (2, 3)	C (2, 4)	D (3, 6)	E (4, 5)	F (5, 6)	Dummy (3, 5)
Critical	yes	yes	yes	-	yes	yes	yes
Free Float (FF)	-	-	-	1	-	-	-

Note that after the previous step, A and E have reached their crash limits while C has 1 day remaining. As there are two critical paths, the possible crashes are shown below:

Activity	B, C	F
Crash Cost/Day ($)	200, 150	200
Remaining Crash Limit (Days)	1, 1	1

Hence, one alternative is to reduce B and C.

reduction limit = min {crash limit, positive FF limit} = min {1, 1} = 1

Thus we can reduce B and C by 1 day each. However, the additional cost per day due to the crashing of B ($200) and C ($150) is $350, which is more than the cost of delay, $170.

The other alternative is to reduce F.

reduction limit = min {crash limit, positive FF limit} = min {1, 1} = 1

Thus we can reduce F by 1 day. However, the additional cost per day due to the crashing of F is $200, which is more than the cost of delay, $170.

Hence, the previous step provides the optimal crashing solution.

Project completion time = 12 days

Cost of delay = (delay time) × (cost of delay/day) = (12 − 10) × 170 = $340.

Project cost = 3550 + 340 = $3890.

14.6 An electrical engineering project has the following activity information:

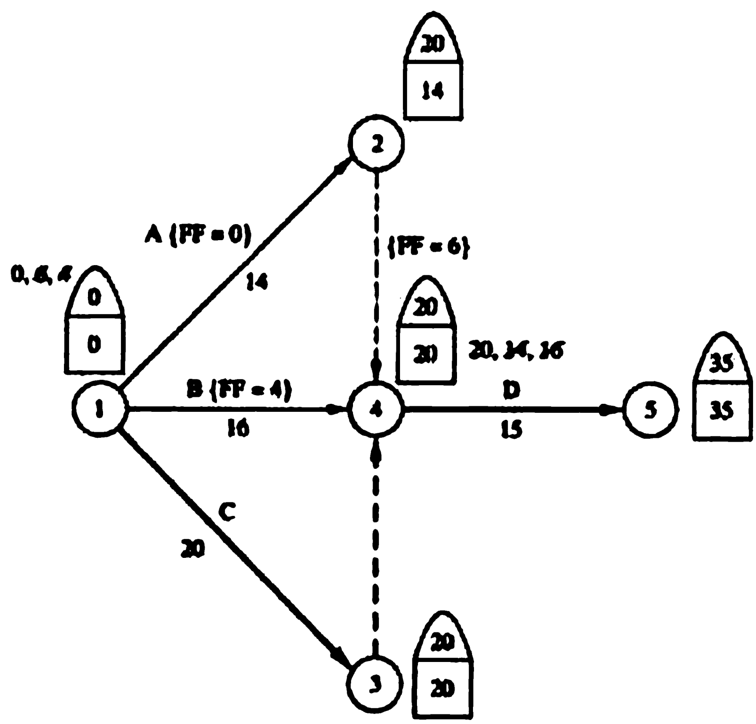
Activity	Immediate Predecessor(s)	Normal Time (Days)	Normal Cost ($)	Crash Time (Days)	Crash Cost ($)
A	-	14	1000	10	1400
B	-	16	1200	11	1650
C	-	20	2000	14	2720
D	A, B, C	15	3000	10	4250

(*a*) Draw the network diagram. Find the critical path, total cost, and total time.

(*b*) If the budget limit is $200 per day for any additional cost due to crashing, find the optimal project completion time and the corresponding cost.

(*c*) If the total budget for this project is $8000 with no limit on daily spending, what is the shortest possible project time?

(*a*)



Critical path is C, D.

Project completion time = 35 days

Project cost = $7200

(*b*) From the given data, construct the following crash time–cost table.

Activity (i, j)	Crash Limit (Days) $= (D - D')$	Crash Cost/Day ($) $= (C' - C)/(D - D')$
A (1, 2)	14 − 10 = 4	(1400 − 1000)/4 = 100
B (1, 4)	16 − 11 = 5	(1650 − 1200)/5 = 90
C (1, 3)	20 − 14 = 6	(2720 − 2000)/6 = 120
D (4, 5)	15 − 10 = 5	(4250 − 3000)/5 = 250

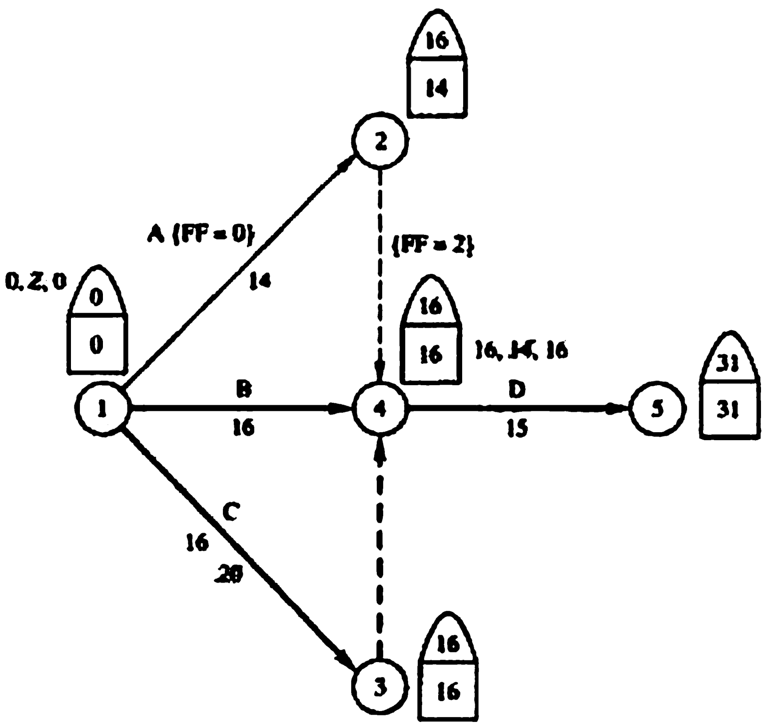
From the critical path calculations, we have the following information.

Activity (i, j)	A (1, 2)	B (1, 4)	C (1, 3)	D (4, 5)	Dummy (2, 4)	Dummy (3, 4)
Critical	-	-	yes	yes	-	yes
Free Float (FF)	0	4	-	-	6	-

Since the critical activity C has the lowest "crash cost per day," it becomes the first candidate for crash. Note that the crash cost per day for C is $120, which is less than the budget limit of $200 per day for any additional cost due to crashing. The length by which C can be reduced is found as follows:

$$\text{reduction limit} = \min\{\text{crash limit, positive FF limit}\} = \min\{6, 4\} = 4$$

Hence, crash activity C by 4 days.



From the critical path calculations, we have the following information.

There are two critical paths: C, D (old) and B, D (new)
Project completion time = 31 days
Project cost = 7200 + (4)(120) = $7680

Activity (i, j)	A (1, 2)	B (1, 4)	C (1, 3)	D (4, 5)	Dummy (2, 4)	Dummy (3, 4)
Critical	-	yes	yes	yes	-	yes
Free Float (FF)	0	-	-	-	2	-

Note that after the previous step, the remaining crash limit for C is 2 days. As there are two critical paths, the possible crashes are shown below:

Activity	B, C	D
Crash Cost/Day ($)	90, 120	250
Remaining Crash Limit (Days)	5, 2	5

Hence one alternative is to reduce B and C.

Reduction limit = min {crash limit, positive FF limit} = min {2, 2} = 2

Thus we can reduce B and C by 2 days each. However, the additional cost per day due to the crashing of B ($90) and C ($120) is $210 which exceeds the budget limit of $200 per day for any additional cost due to crashing.

The other alternative is to reduce D.

Reduction limit = min {crash limit, positive FF limit} = min {5, 2} = 2

Thus, we can reduce D by 2 days. However, the additional cost per day due to the crashing of D is $250 which exceeds the budget limit of $200 per day for any additional cost due to crashing. Hence, the previous step provides the optimal crashing solution.

Project completion time = 31 days

Project cost = $7680

(*c*)

Alternative 1a:
Crash activities B and C by 2 days each.
Project completion time = 29 days
Project cost = 7680 + (2)(210) = $8100 > $8000
Thus, Alternative 1a is infeasible.

Alternative 1b:
Crash activities B and C by one day each.
Project completion time = 30 days
Project cost = 7680 + (1)(210) = $7890 < $8000
Thus, Alternative 1b is feasible.

Alternative 2a:
Crash activity D by 2 days.
Project completion time = 29 days
Project cost = 7680 + (2)(250) = $8180 > $8000
Thus, Alternative 2a is infeasible.

Alternative 2b:
Crash activity D by one day.
Project completion time = 30 days
Project cost = 7680 + (1)(250) = \$7930 < 8000
Thus, Alternative 2b is feasible.

Of the two feasible alternatives 1b and 2b, alternative 1b is optimal, since it has lower project cost. The results of Part (c) are summarized in the following table.

Alternative	1a	1b	2a	2b
Crash Activity (Activities)	B and C	B and C	D	D
Crash Time (Days)	2 days each	1 day each	2 days	1 day
Project Completion Time (Days)	29	30	29	30
Project Cost (\$)	7680+(1)(210) =8100>8000	7680+(2)(210) =7890<8000	7680+(2)(250) =8180>8000	7680+(1)(250) =7930<8000
Feasible	-	yes	-	yes
Optimal	-	yes	-	-

Supplementary Problems

14.7 Develop a network diagram for a project having the following precedence relationships:

Activity	A	B	C	D	E	F	G	H	I	J	K
Immediate Predecessor(s)	-	-	A	A, B	C, D	D	E	E, F	G	H	I, J

14.8 Construct a network diagram for the project consisting of activities A, B, C, . . . , L described below:

Concurrent activities A and B begin the project;
Concurrent activities C and D succeed A;
Concurrent activities E and G succeed B;
Activity F succeeds both C and E;
Activity H succeeds both C and D;
Activities I and J succeed G;
Activity K succeeds H and F;
Activity L succeeds I and J;
Activities L and K complete the project.

14.9 Consider the project in Problem 14.7 with the following activity durations:

Activity	A	B	C	D	E	F	G	H	I	J	K
Duration (Days)	3	2	5	7	3	4	8	13	6	1	10

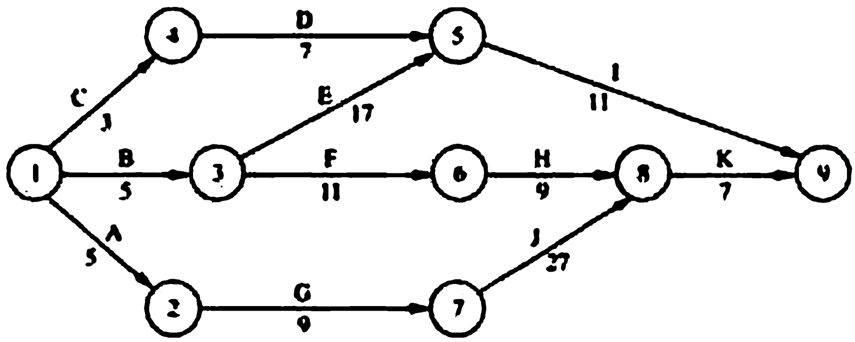
(*a*) Draw a CPM network diagram and find the critical path.

(*b*) The project must be completed in 30 days. Do you anticipate difficulty in meeting the deadline? Explain.

(*c*) Can activity H be delayed without delaying the project?

(*d*) Can activity E be delayed without delaying the project?

14.10 For the following arrow diagram identify the critical path and calculate the total and free floats for each activity.



14.11 The software solution division at Mastek Inc. has been working on an application which on development would have a large market. In order to remain market leaders and innovators of new products, they have to complete this project as soon as possible. The division manager resorts to the use of PERT in the scheduling of the project activities. The following table depicts the information on the activities:

Activity	Immediate Predecessor(s)	Duration (Days) Optimistic (*a*)	Most Likely (*m*)	Pessimistic (*b*)
A	-	2	3	4
B	A	2	4	6
C	A	4	5	12
D	A	1	3	5
E	B	2	2	2
F	B	3	6	9
G	C	5	7	15
H	E, G, D	4	8	12
I	D	6	15	18
J	E, F, G, D	3	4	5

(*a*) Find the critical path and the expected project completion time through a PERT network diagram.

(*b*) What is the probability that the project will be completed within 30 days?

14.12 Consider Problem 14.7. Suppose the activity durations are probabilistic as given in the table below:

Activity	Optimistic (a)	Most Likely (m)	Pessimistic (b)
A	1	3	5
B	1	2	3
C	3	5	13
D	4	7	10
E	2	3	4
F	1	4	13
G	4	8	12
H	6	13	14
I	2	6	10
J	1	1	1
K	9	10	17

(a) Calculate the expected time and variance for each activity.

(b) Find the critical path.

(c) Determine the expected project completion time.

(d) The scheduled completion date for the project is Feb. 5. If you plan to start the project on Jan. 1, find the probability that you will complete the project by then. Should you start the project earlier?

14.13 Consider a construction project with the following data on precedence relationships, durations, and costs:

Activity	Immediate Predecessor(s)	Normal Time (Days)	Normal Cost ($)	Crash Time (Days)	Crash Cost ($)
A	-	6	120	4	170
B	-	4	120	2	220
C	A	3	195	2	270
D	A	4	320	2	520
E	B, C	7	700	4	1075
F	D, E	5	650	2	1100
G	E	10	1600	6	2300

F and G are the terminal activities of the project.

(a) Find the critical path.

(b) Find the project completion time and the corresponding cost.

(c) Suppose it is required to complete the project in 22 days. Find which activities to crash and by how much, to yield the minimum project cost.

14.14 Consider the following information for a manufacturing systems project:

Activity	Immediate Predecessor(s)	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost ($)	Crash Cost ($)
A		12	10	50 000	90 000
B	A	10	8	140 000	170 000
C	B	12	9	120 000	180 000
D	A	9	8	60 000	70 000
E	D	12	10	70 000	95 000
F	C, E	5	5	80 000	80 000
G	F	6	6	60 000	60 000

(a) Draw the network diagram and find the critical path.

(b) Find the project completion time and the corresponding cost.

(c) If the company wants to complete the project in 41 weeks, find the optimal crash time and cost.